CHAPTER

11

SECOND-ORDER SECTION

We have done integration and differentiation with simple, single-time-
constant circuits that had $\tau s + 1$ in the denominator of their transfer
functions. These systems gave an exponentially damped response to step
or impulse inputs. In Chapter 8, we showed how a second-order system
can give rise to a sinusoidal response. In this chapter, we will discuss a
simple circuit that can generate a sinusoidal response. We call this circuit
the \textbf{second-order section}; we can use it to generate any response that
can be represented by two poles in the complex plane, where the two poles
have both real and imaginary parts. With this circuit, we can adjust the
positions of the complex-conjugate poles anywhere in the plane.

The second-order circuit is shown in Figure 11.1; it contains two cas-
caded follower–integrator circuits and an extra amplifier. The capacitance $C$
is the same for both stages ($C_1 = C_2 = C$), and the transcond-
uctance of the two feed-forward amplifiers, $A1$ and $A2$, are the same:
$G_1 = G_2 = G$ (approximately—if $G$ is defined as the average of $G_1$ and
$G_2$, small differences will have no first-order effect on the parameters of
the response). We obtain an oscillatory response by adding the feedback
amplifier $A3$. This amplifier has transconductance $G_3$, and its output
current is proportional to the difference between $V_2$ and $V_3$, but the sign
of the feedback is \textit{positive}; for small signals, $I_3$ is equal to $G_3(V_2 - V_3)$.

If we reduce the feedback to zero by shutting off the bias current
in $A3$, each follower–integrator circuit will have the transfer function given
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FIGURE 11.1  Circuit diagram of the second-order section. The amplifier A3 tends to keep $V_2$ ahead of $V_1$, once $V_2$ has gained the lead. For that reason, A3 has a destabilizing effect on the circuit behavior.

in Equation 9.3 (p. 148). Two follower–integrator circuits in cascade give an overall transfer function that is the product of the individual transfer functions, so there are two poles at $s = -1/\tau$:

$$\frac{V_3}{V_1} = \left(\frac{1}{\tau s + 1}\right)^2 = \frac{1}{\tau^2 s^2 + 2\tau s + 1}$$

We can understand the contribution of A3 to the response by following through the dynamics of the system when a perturbation is applied to the input. Suppose we begin with the input biased to some quiescent voltage level. In the steady state, all three voltages will settle down, and $V_2$ and $V_3$ will both be equal to $V_1$. If we apply to $V_1$ a small step function on top of this DC level, $V_2$ starts increasing, because we are charging up the first capacitor $C_1$. Eventually, $V_2$ gets a little ahead of $V_3$, and then amplifier A3 makes $V_2$ increase even faster. Once $V_2$ is increasing, the action of A3 is to keep it increasing; the feedback around the loop is positive. If we set the transconductance $G_3$ of amplifier A3 high enough, $V_2$ will increase too fast, and the circuit will become unstable.

SMALL-SIGNAL ANALYSIS

A2 is a follower–integrator circuit, so, from Equation 9.3 (p. 148),

$$V_3 = \frac{V_2}{\tau s + 1}$$  \hspace{1cm} (11.1)

where $\tau = C/G_2$. The current $I_1$ coming out of amplifier A1 is proportional to the difference between $V_1$ and $V_2$, its two inputs:

$$I_1 = G_1(V_1 - V_2)$$  \hspace{1cm} (11.2)

We can describe $I_3$, the output of A3, in the same way:

$$I_3 = G_3(V_2 - V_3)$$  \hspace{1cm} (11.3)
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We need an equation for $V_3$ in terms of $V_1$. Combining the two currents into the capacitor (Equations 11.2 and 11.3), we obtain

$$C \frac{dV_2}{dt} = G_1(V_1 - V_2) + G_3(V_2 - V_3)$$

Using $s = d/dt$ and collecting terms,

$$V_2(sC + G_1 - G_3) = G_1 V_1 - G_3 V_3$$

Substituting $V_2$ from Equation 11.1, and simplifying using $\tau = C/G_1 = C/G_2$ and $\alpha = G_3/(G_1 + G_2)$,

$$H(s) = \frac{V_3}{V_1} = \frac{1}{\tau^2 s^2 + 2\tau s(1-\alpha) + 1} \quad (11.4)$$

Equation 11.4 is the transfer function for the circuit; as we expected, it is a second-order expression in $\tau s$. The parameter $\alpha$ is the ratio of the feedback transconductance $G_3$ to the total forward transconductance $G_1 + G_2$. If $\alpha$ is equal to 0, Equation 11.4 should give the response of two first-order sections. The denominator is $(\tau s + 1)^2$, just as we expected.

We also can see that, when $\alpha$ is equal to 1, the center term in the denominator becomes 0, and we get

$$\frac{V_3}{V_1} = \frac{1}{\tau^2 s^2 + 1}$$

Under these conditions, the roots of the natural system response are thus

$$\tau^2 s^2 = -1 \quad \text{or} \quad \tau s = \pm j$$

### Complex Roots

We can put the poles on the imaginary axis when $\alpha$ is equal to 1, or right down on the real axis when $\alpha$ is equal to 0. Now our job is to determine

1. Where the poles are located when $\alpha$ is neither 0 nor 1
2. How the system responds under such conditions

We can write the transfer function

$$\frac{V_3}{V_1} = \frac{1}{(\tau s - \tau R_1)(\tau s - \tau R_2)} \quad (11.5)$$

where $R_1$ and $R_2$ are the roots of the denominator in the s-plane. We can define the position of any root as $Re^{j\theta}$, where $R$ is the distance to the root from the origin and $\theta$ is the angle from the positive real axis to the root; that is just the polar form of a complex number. As we discussed in Chapter 8, complex roots of real polynomials must occur as complex-conjugate pairs:

$$R_1 = Re^{j\theta} \quad \text{and} \quad R_2 = Re^{-j\theta} \quad (11.6)$$
We will solve for $R$ and $\theta$ in terms of $\alpha$ and $\tau$ by comparing the denominator of Equation 11.5 with that of Equation 11.4. Substituting Equation 11.6 into Equation 11.5, we obtain

$$\tau^2 s^2 + 2\tau s(1 - \alpha) + 1 = \tau^2 s^2 - \tau^2 sR(e^{j\theta} + e^{-j\theta}) + \tau^2 R^2$$

Using the identity $2\cos\theta = e^{j\theta} + e^{-j\theta}$, we arrive at an important result:

$$\tau = \frac{1}{R} \quad \text{and} \quad -\cos\theta = 1 - \alpha$$

The roots are located on a circle of radius $1/\tau$; they move to the right from the negative real axis as we increase the transconductance of the feedback amplifier. The angle of the roots is determined by the ratio of feedback transconductance to forward transconductance, and is independent of the absolute value of $\tau$.

We can normalize the plot, so the roots lie on the unit circle, by expressing all distances in the plane in units of $1/\tau$, as shown in Figure 11.2. From this construction, we have the following important relation between the major circuit variables:

$$(\omega\tau)^2 + (\sigma\tau)^2 = 1 \quad (11.7)$$

We can see from Figure 11.2 that $\cos\theta$ is the projection of either root onto the $\sigma$ axis. The poles are $1 - \alpha$ to the left of the origin. The real part $\sigma$ of both roots is given by

$$-\sigma\tau = 1 - \alpha \quad (11.8)$$

When $\alpha$ is 1, the real part is 0, and we are left with a pair of roots on the $\pm j\omega$ axis. When $\alpha$ is 0, the root pair is at the point $-1$ on the negative real axis; for $\alpha$ greater than 0, the distance between that point and the real part of the roots is just $\alpha$.

We now have a way to visualize the effect of the feedback ratio $\alpha$ on the location of the roots: $\alpha$ is the horizontal distance from the roots to their original

**FIGURE 11.2** Complex-plane representation of small-signal behavior of the second-order section. The effect of $\alpha = G_3/(G_1 + G_2)$ is to move the poles toward the $j\omega$ axis. For $\alpha$ greater than 1, the circuit is unstable.
position when there was no feedback in the circuit. The roots start at $-\frac{1}{\alpha}$ on the real axis; as we increase $G_3$, we push them to the right, decreasing the magnitude of the damping constant $\sigma$. Eventually, we push them across the $j\omega$ axis and the circuit begins to oscillate. We have shown that we have an independent control on the location of the roots on the circle. The radius of the circle is determined by $G$. By changing $\alpha$ to be equal to $G_3/(2G)$, we can locate the roots anywhere on the circle. As we change the location of the roots, we change the response of the circuit. That is why the second-order section is such a useful device.

Second-order systems often are characterized in terms of a $Q$ parameter, defined by $Q = -1/(2\sigma \tau)$, or by the transfer function expression

$$H(s) = \frac{1}{\tau^2 s^2 + \frac{1}{4} \tau s + 1}$$

By comparing this expression with Equation 11.4 or Equation 11.8, we find that

$$Q = \frac{1}{2(1 - \alpha)}$$

Note that $Q$ starts from 0.5 with no feedback ($\alpha = 0$), and grows without bound as the feedback gain approaches the total forward gain ($\alpha = 1$ or $G_3 = G_1 + G_2$); beyond this point, small signals grow exponentially—the circuit is unstable.

**Transient Response**

In Chapter 8, we saw that the natural response of a linear system always could be written

$$V(t) = e^{st} = e^{\sigma t}e^{j\omega t}$$

where the value of $s$ is given by the root or conjugate pair of roots of the denominator of the transfer function. The impulse response of the circuit, for positive $t$, is of the same form as this natural response.

Depending on the value of $\alpha$, the behavior of the second-order section may be best described in either the time domain or the frequency domain. The impulse response of the circuit when the roots are on the real axis is just an exponential; the response does not oscillate at all, and there is therefore no frequency associated with it. It is simply a dying exponential. When the roots are on the imaginary axis, the circuit is an oscillator—it just sits there and oscillates, on and on and on. For $\alpha$ between 0 and 1, the impulse response is a damped sine wave, as shown in Figure 11.3, because $s$ has both a real and an imaginary component. The $e^{j\omega t}$ is an oscillating response, composed of sines and cosines. The $e^{\sigma t}$ is the damping term, as long as $\sigma$ is negative.

The values of $\omega$ and $\sigma$ can be determined directly from Figure 11.3. The duration of one cycle is 460 microseconds. A cycle is $2\pi$ radians; $\omega$ is thus equal to $1.37 \times 10^4$ radians per second, which is about $1/\tau$. The wave damps by a factor of $1/e$ in 2.6 milliseconds. The damping constant $\sigma$ is thus $-3.85 \times 10^2$ per second.
The value of $\alpha$ from Equation 11.8 is

$$\alpha = 1 + \sigma \tau = 0.97$$

The circuit becomes unstable when $G_3$ is greater than $2G$. We can think about the onset of instability in the following way. There are two amplifiers (A1 and A2) with negative feedback, but there is only one amplifier (A3) with positive feedback. Negative feedback damps the response, and positive feedback reduces the damping. To make the circuit unstable, A3 must provide as much current as do the two amplifiers, A1 and A2, that provide the damping. When the two effects are equal, the circuit is just marginally damped. As we increase $G_3$ above $2G$, the damping becomes negative, and the response becomes an exponentially growing sine wave. Exponential growth is an explosive kind of thing, so the second-order section rapidly leaves the small-signal regime, and becomes dominated by large-signal effects, as we discuss later in this chapter.

**Frequency Response**

When the damping of the circuit is low, we find it natural to view the response as a function of frequency. The frequency response of the second-order section of Figure 11.1 is shown in Figure 11.4 for a number of values of $\alpha$. The highest peak corresponds to the setting used for the transient response shown in Figure 11.3.

We can evaluate the frequency response by substituting $j\omega$ for $s$ into Equation 11.4:

$$\frac{V_3}{V_1} = \frac{1}{-\omega^2 \tau^2 + j2\omega \tau (1 - \alpha) + 1}$$

We can simplify the algebra by computing $D$, the magnitude of the denominator of the transfer function, in terms of a normalized frequency $f = \omega \tau$ and...
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FIGURE 11.4  Measured frequency response of the second-order section for several values of \( \alpha \). The highest curve is for \( \alpha \approx 0.94 \), slightly lower than the value used for the transient behavior shown in Figure 11.3. The lowest curve is for \( \alpha = 0 \), and the second curve is very near \( Q^2 = 1/2 \).

\[
2(1 - \alpha) = 1/Q. \text{ Using the fact that the magnitude of a complex number can be computed from the Pythagorean theorem, we have}
\]

\[
D^2 = (1 - f^2)^2 + \frac{f^2}{Q^2} = f^4 - f^2 \left( 2 - \frac{1}{Q^2} \right) + 1 \quad (11.9)
\]

It is convenient to plot the log of the magnitude of the transfer function as a function of log \( f \). But

\[
\log \left| \frac{V_3}{V_1} \right| = -\frac{1}{2} \log(D^2)
\]

Hence, we can reason about the response directly from the behavior of \( D^2 \). When \( f \) is small, the \( f^4 \) term is much smaller than is the \( f^2 \) term, so

\[
\log \left| \frac{V_3}{V_1} \right| = -\frac{1}{2} \log \left( 1 - f^2 \left( 2 - \frac{1}{Q^2} \right) \right) \approx f^2 \left( 1 - \frac{1}{2Q^2} \right)
\]

At low frequencies (\( f \) much less than 1), the response grows larger as \( f \) is increased, provided \( Q^2 \) is greater than 1/2 or \( Q \) is greater than 0.707. At some frequency, the \( f^4 \) term is no longer negligible, and it starts canceling the effect of the \( f^2 \) term. Above that frequency, \( f^4 \) increases much faster than \( f^2 \) does, so the response decreases. Eventually, the \( f^4 \) term is much larger than the other terms are, so the response decreases as \( 1/f^2 \), because

\[
-\frac{1}{2} \log(f^4) = -2 \log f
\]

The plot decreases with a slope of \(-2\) when \( f \) is much greater than 1.

So we know the asymptotes. Near zero frequency, the gain is 1 and the response is flat (independent of frequency); at very high frequencies, the slope on a log scale approaches \(-2\), because the response is proportional to \( 1/f^2 \); between
the two extremes, there is a maximum. The \( f^2 \) term increases before the \( f^4 \) term does; as \( f^2 \) increases, so does the response. Eventually, the \( f^4 \) term becomes large enough to dominate the \( f^2 \) term, and then the response begins to decrease.

The response will be a maximum where \( D^2 \) is a minimum—that is, where the derivative is zero:

\[
\frac{dD^2}{df} = 4f^3 - 2f \left( 2 - \frac{1}{Q^2} \right) = 0
\]

\[
f^2_{\text{max}} = 1 - \frac{1}{2Q^2}
\]

Equation 11.10 tells us where the peak in the response curve is. Now we can take that maximum frequency from Equation 11.10 and put it back into the transfer function Equation 11.9 to find the value of the denominator at the peak:

\[
D^2_{\text{max}} = \frac{1}{Q^2} \left( 1 - \frac{1}{4Q^2} \right)
\]

Equation 11.11 gives a maximum value of the transfer function

\[
\left| \frac{V_3}{V_1} \right|_{\text{max}} = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}
\]

So, as \( Q \) becomes large, the height of the peak approaches \( Q \), and the peak frequency approaches \( 1/\tau \).

When \( Q^2 \) is equal to \( 1/2 \), the peak gain is 1 at zero frequency and the response is maximally flat (that is, the lowest-order frequency dependence is \( f^4 \)). For lower \( Q \) values, the gain drops off quadratically with frequency.

**LARGE-SIGNAL BEHAVIOR**

Thus far, we have been concerned with the second-order section as a linear system. The linear approximation is valid for small amplitudes of oscillation. As we might expect, the circuit has all the slew-rate limitations we saw for first-order filters. When the second-order section becomes slew-rate limited, however, its behavior is much more exciting than that of its first-order cousins. When the circuit that generated the small-signal impulse response of Figure 11.3 is subjected to a large impulse input, it breaks into a sustained limit-cycle oscillation, as shown in Figure 11.5. The amplitude of the oscillation is the full range of the power supply. Thus, the circuit that is perfectly stable for small signals becomes wildly unstable for large signals. We need a little imagination to visualize a system controlled by such a circuit, gripped by recurring seizures of this violent electronic epilepsy. As with any pathology, we must understand the etiology, and take precautions against any possible onset of the disease.

We can analyze this grotesque behavior by realizing that the input voltages to all three amplifiers are many \( kT/(q\kappa) \) units apart over almost all parts of the waveform. Under these conditions, the currents out of the amplifiers are constant,